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Monotonic System Framework for Globally Optimal Structuring of Complex Data

Fatima T. Adilova[a,*], Rifqat R. Davronov[a] and Yalkin T. Adilov[b]

[a] *Laboratory of Medical and Biological Informatics, V.I. Romanovsky Institute of Mathematics, Academy of Sciences of Uzbekistan, 9 University Street, 100174 Tashkent, Uzbekistan*
[b] *Tashkent Institute of Irrigation and Agricultural Mechanization Engineers — National Research University*

fatadilova@matinst.uz

Abstract

The structuring and partitioning of large-scale complex datasets is often limited by the computational complexity of combinatorial optimization, which frequently yields suboptimal, locally extremal solutions. This paper proposes a structured processing methodology based on monotone systems (MS) theory that reformulates structural identification as a globally optimal optimization problem. We analyze the mathematical behavior of element–subset weight functions $\pi(i, H)$ and extend classification through the concept of associative patterns. A functional software architecture for generating monotonic systems is developed and evaluated on two tasks: classification of handwritten digit images from the scikit-learn digits dataset (1797 images, 10 classes) and clinical stratification of patients with ischemic heart disease (IHD). In the image experiment, we compare weight functions of different types: the second-type function (6) with internal parametrization coefficient α achieves maximum baseline accuracy at $\alpha = 0.6$–0.9, while the first-type function (5) with power parameter $\delta = 2$ combined with an ensemble of random column partitions reaches 97.4% classification accuracy on the full 10-class problem. In the cardiology case study based on 62 patients with acute myocardial infarction, core extraction with a first-type connection function produces a stability index $g = 0.75$, supporting robust patient grouping. The results demonstrate that MS-based core extraction provides interpretable, globally optimal structure identification for heterogeneous object–feature data.

Keywords: Monotonic systems; Data aggregation; Core extraction; Associative patterns; Classification

1 Introduction

In the rapidly evolving domain of artificial intelligence in medicine, the efficient automated structuring of heterogeneous data remains a significant challenge. Modern diagnostic workflows generate vast amounts of complex data, ranging from raw pixel matrices in digital radiography, CT, and MRI scans to unstructured clinical matrices extracted from electronic health records (EHR). Traditional deep learning and heuristic clustering methods frequently suffer from the limitations of local extrema, semantic opacity, and a high sensitivity to noise in blurred or low-contrast images. By transforming these combinatorial problems into exact, globally optimal tasks, the theory of Monotonic Systems (MS) offers a powerful alternative. Specifically, the mathematical extraction of a system's core enables the identification of robust, low-dimensional associative images within high-dimensional matrices.

At present, in identifying the structure of complex systems, there is significant interest in approaches where information is represented in the form of a graph or a data matrix. Representing a system as a single entity—composed of elements and the connections between them—lends meaning to the task of system structure identification. The structure of a system can be formulated through a certain

*Corresponding author.

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set of elements $P_1, \ldots, P_M$ of arbitrary nature (objects, features) defined by a specific data matrix. For each pair of elements P_i and P_j, a value α_{ij} is assigned, which is interpreted as the strength of the connection or the degree of proximity between them. The set of elements $P_1, \ldots, P_M$ is characterized by the connection matrix:

$$A = \{\alpha_{ij}\}_{i,j=1}^{M}$$

The objective of processing the connection matrix is to partition the entire set of elements into non-overlapping subsets $G_1, \ldots, G_L$, where the connection values α_{ij} between elements within the same subset are greater than those between elements belonging to different subsets. Under fairly general assumptions regarding the object-feature system, it is possible to construct a data analysis method that ensures the global extremum of the objective function is obtained across all possible system partitions. To achieve this, a numerical function is defined to measure the importance of each system element; as the importance of a single element changes, this function monotonically alters the importance of other elements in the system. For such systems, a theory of core extraction was formulated in (Mullat, 1976; Kuznetsov et al., 1985; Mirkin and Muchnik, 2002; Muchnik and Schwarzer, 2010; Muchnik et al., 1986; Makdesi et al., 2024), which allows for the construction of various monotonic systems of elements on a data matrix, thereby enhancing interpretative capabilities, for instance, in solving classification problems. The monotonicity property of systems leads to the concept of the system core as a subsystem that reflects the structure of the entire system as a whole in terms of element connections. The theory of monotonic systems (MS) in solving structuring problems removes the limitations of traditional methods at the problem statement level. Therefore, applying MS theory to such problems reduces to the typical task of describing the structure of a large system using the language of a small system—the core.

The aim of this work is to study the properties of structural data processing methods based on MS theory and to investigate their interpretability and analogy to associative thinking processes (Mirkin and Muchnik, 2002; Muchnik and Schwarzer, 2010). The study examines the properties of the weight function $\pi(i, H)$, modifies the classification problem using the concept of an associative pattern, and presents a functional software architecture for generating monotonic systems. The practical contribution is illustrated through two case studies: classification of single-digit images and classification of patients with ischemic heart disease (IHD).

2 Material and Methods

2.1 Dataset

The image classification study was conducted on the scikit-learn handwritten digits dataset (Pedregosa et al., 2011), which contains 1797 grayscale images of digits 0–9. Each image is an 8×8 pixel matrix with integer intensity values in the range $[0, 16]$. The dataset was split into a training set of 1257 images and a test set of 540 images (70/30 stratified split). Experiments were performed in two settings: a three-class problem (digits 1, 3, and 5; $n_{train} = 382$, $n_{test} = 165$) and a full ten-class problem (digits 0–9). As a practical case, data from the Republican Institute of Cardiology were analyzed, comprising 62 patients with acute myocardial infarction (men aged 30 to 65 years). The diagnosis of acute myocardial infarction (AMI) was established in accordance with the criteria proposed by WHO experts.

2.2 Ethics statement

The clinical hemodynamic data analyzed in this study were originally collected within prior clinical research at the Republican Institute of Cardiology (Kamilova, 1991). This work presents a secondary mathematical analysis of anonymized patient records. Individual patient identifiers were not used during processing. The study was conducted in accordance with the Declaration of Helsinki and applicable institutional ethical standards governing retrospective analysis of clinical data.

2.3 Methods

We present the problem of generating a family of monotonic systems using the analysis of "object-feature" Boolean matrices as an example. The construction of monotonic systems consists of the following steps:

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- construction of a set of transformation operators for monotonic systems of a fairly general form, defined on the same abstract finite set W, where $|W| = N$;
- construction of a set of base functions on the selected specific sets W.

2.3.1 Construction of a Set of Transformation Operators for Monotonic Systems

In (Muchnik et al., 1986), three theorems were proven that define the MS generation system as a structure creating compositional chains of operators over selected base MSs. Obviously, by changing the sets of operators and base systems, one can obtain different MS generation systems.

2.3.2 Construction of a Set of Base Functions

The core element of an MS, the variation of which provides mechanisms for generating a large number of different MSs, is the family of weight functions $\pi(i, H)$. These functions determine the degree of connection (influence, similarity) between element i and the subset H containing this element for all possible pairs (i, H), where $i \in H \subseteq W$.

Weight functions of the first type measure the connection between an element and a set as the sum of pairwise connections between the given element and each element of the set:

$$\pi(i, H) = \sum_{k \in H} a_{ik}, \quad i \in H,\ H \subseteq W \tag{1}$$

where a_{ik} is a certain measure of the pairwise connection between the i-th and k-th elements, which can be defined as a connection matrix $A = \| \ a_{ik} \ \|_N^N$ calculated from the original matrix Φ (it is assumed that $a_{ii} = 0$).

Components of the second type of MS are identified by specifying a representative element of an arbitrary subset and the measure of connection between the given i-th element and this representative. Two variants of the representative element are known (Kuznetsov et al., 1984), defined by the feature sets Y^H and Y_H:

$$Y^H = \bigcap_{k \in H} y_k, \quad Y_H = \bigcup_{k \in H} y_k \tag{2}$$

The elements φ_{Hj}, $j = \overline{1, N}$ of the corresponding constructed Boolean rows are defined by the following relations:

$$\varphi_{Hj} = \min_{k \in H} \varphi_{kj}, \quad \varphi_{Hj} = \max_{k \in H} \varphi_{kj}$$

The measure of connection between element i of set H and the representative i_H of this set is defined in the first case as:

$$\pi_1(i, H) = a^1_{ii_H} = |y_i \setminus Y^H| = |y_i| - |Y^H| \tag{3}$$

and in the second case as:

$$\pi_2(i, H) = a^2_{ij_H} = |Y_H \setminus y_i| = |Y_H| - |y_i|$$

Weight functions of the third type measure the connection between an element and a subset as the sum of the connections between the element and the group for each individual feature. In this case, each feature $j \in Y$ is characterized by a certain value $\omega_j(H)$ that depends on the set H, and these values are then summed over all features y_i of the given element i:

$$\pi(i, H) = \sum_{j \in y_i} \omega_j(H), \quad \forall i \in H,\ H \subseteq W \tag{4}$$

Furthermore, the function $\omega_j(H)$ used must be monotonic:

$$\omega_j(E) \leq \omega_j(H), \quad \forall E \subseteq H \subseteq W, \forall j \in Y$$

The coefficients of the pairwise connection between Boolean vectors (or subsets of a certain set) can vary. It is precisely the set-theoretic operations used when selecting the pairwise connection coefficient of objects that allow for altering the substantive interpretation of the core and other extracted subsets of the first type of MS.

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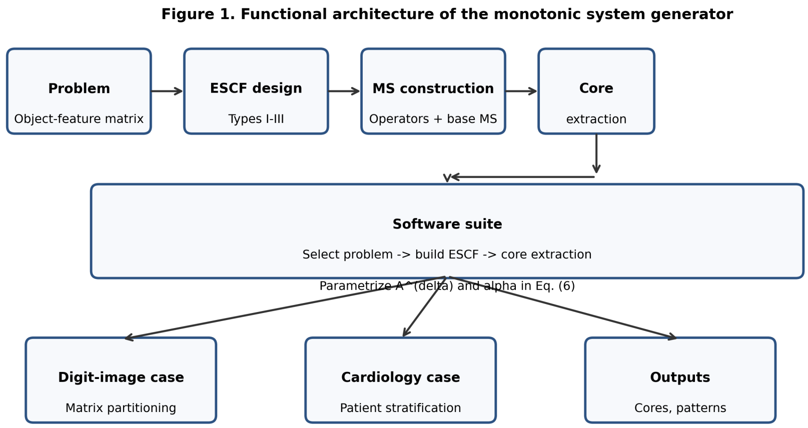

Figure 1: Functional architecture of the monotonic system generator: problem specification, ESCF construction, core extraction, and application to the two case studies.

All the methods discussed above for generating parametric and compositional families of MS represent external parametrization, in which a new element-subset connection function is constructed from base functions. We obtain internal parametrization of individual components of base weight functions in an MS system for functions of the first type (1) if, instead of the original connection matrix $A = \| a_{ik} \|_N^N$, we use the matrix $A^{([\delta])} = \| a_{ik}^{\delta} \|_N^N$, where δ is a non-negative power parameter, and:

$$\pi(i, H) = \sum_{k \in H} a_{ik}^{\delta}, \quad i \in H,\ H \subseteq W \tag{5}$$

To develop and construct a methodology for analyzing specific systems, it is necessary to use those MSs whose computational properties are most thoroughly studied. Such a class of monotonic systems is provided by the internal parametrization of systems (3) and (4). In (Adilova and Davronov, 2026a), we developed this approach within the framework of the class of evaluation estimation algorithms (Zhuravlev et al., 1974).

3 Computational Experiments

The primary difficulty in applying MS theory in practice lies in the lack of tools for generating "element-subset" connection functions (ESCF). Therefore, the MS generation system is implemented as a software suite (Adilova and Davronov, 2026a;b), the operation of which involves a preliminary analysis of the complex system structuring problem statement and the selection of a path to solve this problem. The user interface allows defining the type of problem to be solved; based on this, a problem-solving scheme is constructed, which involves either selecting or constructing the corresponding ESCF and choosing an algorithm for MS core extraction. The overall workflow is summarized in Figure 1.

3.1 Solving the Image Classification Problem

The classification algorithm involves applying a simultaneous partitioning to the original data matrix, dividing its columns (pixel features) into a predefined number k of submatrices and partitioning its rows (samples) into two classes within each of the k column submatrices. Within each submatrix, the partitioning is implemented by introducing an MS with the weight function:

$$\pi^i(r, H) = \alpha |Y_H^i| - (1 - \alpha)|y_r^i| \tag{6}$$

and finding its corresponding extremal subsets G and $\overline{G}$. Each sample receives a k-bit binary signature encoding its group membership across all column submatrices. Classification is then performed by comparing test signatures against training prototypes using nearest-neighbor matching.

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Table 1: Effect of the internal parametrization coefficient α on three-class digit classification accuracy using weight function (6) with $k = 8$ spatial column partitions.

α	Correctly classified / 165	Accuracy (%)
0.0	60	36.4
0.1	60	36.4
0.2	60	36.4
0.3	64	38.8
0.4	118	71.5
0.5	116	70.3
0.6	126	76.4
0.7	126	76.4
0.8	126	76.4
0.9	126	76.4
1.0	55	33.3

Table 2: Comparison of MS-based classification strategies on the scikit-learn digits dataset.

Method	Weight function	k	Encoding	Acc. (1,3,5)	Acc. (0–9)
Single partition	(6), $\alpha = 0.8$	8	binary	76.4%	—
Single partition	(5), $\delta = 2$	6	grayscale	84.8%	30.4%
Ensemble ($N = 30$)	(5), $\delta = 2$	6	grayscale	99.4%	97.4%

3.1.1 Baseline experiment with weight function (6)

The experiment was performed on the scikit-learn digits dataset ($n = 1797$ images, 8×8 pixels). The pixel matrix was binarized at threshold $t = 4$, and the 64 feature columns were partitioned spatially into $k = 8$ groups corresponding to contiguous image columns. The coefficient α in (6) was varied over the grid $\alpha \in \{0.0, 0.1, \ldots, 1.0\}$. Table 1 summarizes the classification accuracy on the three-class subproblem (digits 1, 3, and 5; $n_{test} = 165$).

The highest classification accuracy of 76.4% was achieved at $\alpha = 0.6$–0.9, confirming the sensitivity of weight function (6) to the parametrization coefficient: low values of α (≤ 0.3) produce trivial partitions, while $\alpha = 1.0$ degenerates to a constant function. The optimal range $\alpha \in [0.6, 0.9]$ balances the contribution of the group representative $|Y_H|$ and the individual row weight $|y_r|$.

3.1.2 Comparison with weight function (5)

To evaluate the effect of the connection function type, we replaced the second-type weight function (6) with the first-type function (5) using the internally parametrized connection matrix $A^{(\delta)} = \| a_{ik}^{\delta} \|$, where pairwise connections a_{ik} are computed as dot products of sample feature vectors and $\delta = 2$. This function does not depend on the external parameter α; instead, the power parameter δ controls the selectivity of pairwise connections, amplifying strong connections while suppressing weak ones.

With a single partition ($k = 6$ contiguous column groups, grayscale pixel values normalized to $[0, 1]$), function (5) achieves 84.8% accuracy on the three-class problem — an improvement of 8.4 percentage points over the best result with function (6).

3.1.3 Ensemble classification and scaling to ten classes

To further increase discriminative power, we employ an ensemble of $N = 30$ independent random column permutations, each partitioned into $k = 6$ groups. Each ensemble member produces a k-bit signature via core extraction using function (5); the N signatures are concatenated into a 180-bit descriptor and classified by 1-nearest-neighbor matching. Table 2 summarizes the results.

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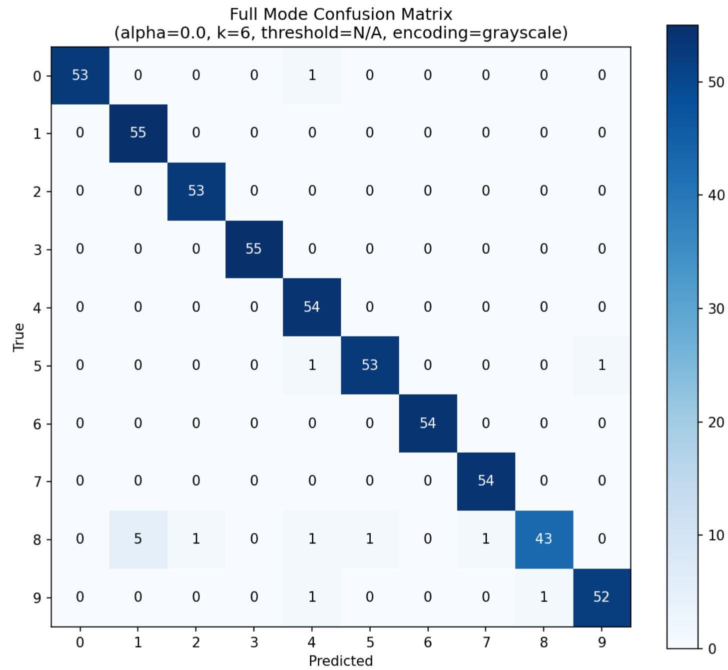

Figure 2: Confusion matrix of the ensemble MS classifier on the full 10-class digit recognition problem (accuracy 97.4%).

Table 3: Summary of the cardiology case study based on 62 AMI patients.

Item	Value / description
Number of patients	62
Selected hemodynamic variables	6 of 12
Connection function	First type (Euclidean metric)
Extracted patient groups	4 (WHO-based clinical classes)
Core stability index g	0.75

The ensemble strategy dramatically improves classification accuracy, achieving 99.4% on the three-class problem (164/165 correctly classified) and 97.4% on the full ten-class problem (526/540 correctly classified). Figure 2 shows the confusion matrix for the ten-class experiment; the majority of errors involve digit 8, which is occasionally misclassified as 1.

3.2 Identifying Types of IHD Development

The objective was to study the impact of the combined administration of phosphocreatine and corinfar in treating patients with complications in the form of acute myocardial infarction (Kamilova, 1991). Out of 12 central hemodynamic parameters measured in a patient, six were selected. The proximity between objects (patients) was measured using the Euclidean metric; a connection function of the first type was utilized. At the output for the compared groups of objects (cores), a stability index $g = 0.75$ was obtained (Table 3), where g quantifies the relative tightness of the extracted patient cores with respect to the optimal MS partition. Values close to 1 indicate highly stable grouping of clinically similar hemodynamic profiles. This result confirmed the validity of the hypothesis and the problem-solving methods during discussions with medical experts.

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4 Discussion

In the image classification problem, an associative-structural method was applied, as it relies on the preliminary extraction of a set of associative images for each object. A crucial element of the proposed method was the use of the MS method, which ensures that a globally optimal solution is obtained. The experiments on the scikit-learn digits dataset demonstrate that the MS framework scales effectively from small toy examples to realistic image classification tasks involving 1797 samples and 10 classes. The comparison of weight functions reveals two complementary roles of internal parametrization: the coefficient α in function (6) controls the balance between group and individual contributions to the partition, while the power parameter δ in function (5) amplifies discriminative pairwise connections. The ensemble strategy — aggregating 30 independent random column partitions — further boosts accuracy from 84.8% to 99.4% on the three-class problem and achieves 97.4% on the full ten-class problem, confirming that the MS-based partition signatures capture robust structural invariants of the digit images.

In a cardiology case study, the conventional division into four groups was made in accordance with the WHO classification; however, in many cases, it fails to account for individual variations in the course of the disease, and thus remains arbitrary. Our findings do not contradict medical practice; on the contrary, they provide attending physicians with additional information regarding prescriptions and treatment strategies. From the perspective of this study, the results confirm the potential of MS in identifying an associative image as a link in logical thinking in general and in clinical reasoning in particular.

5 Conclusion

The problem of structuring large datasets is typically formulated as a combinatorial optimization problem; therefore, its solution can only be approximate when heuristic methods are used. Traditional heuristic structuring algorithms usually employ one or two free parameters for tuning to a specific data matrix, but the resulting solution remains only locally optimal. The theory of monotonic systems can be applied to obtain exact solutions to structuring problems. The distinct feature of the data processing approach within the MS framework is that an element of the data matrix reflects the mutual influence between an object and a feature. Utilizing the concept of an associative pattern has made it possible to formulate the classification problem as a sequential procedure for aggregating "object–associative image" matrices. The experiment in cardiology involving the extracted associative patterns over the original "object–feature" matrix demonstrated their identity.

The image classification experiments on the scikit-learn digits dataset further demonstrate the practical scalability of the approach. A single MS partition with weight function (6) achieves 76.4% accuracy on a three-class problem, with the internal parametrization coefficient $\alpha \in [0.6, 0.9]$ yielding the best results. Replacing function (6) with the first-type weight function (5) using $\delta = 2$ improves single-partition accuracy to 84.8%. An ensemble of 30 random column partitions with function (5) achieves 99.4% on the three-class problem and 97.4% on the full ten-class problem, establishing MS-based partition signatures as a competitive classification method for image data.

Consequently, it is possible to consider a potential formal representation of logical reasoning through the aggregation procedures of specially constructed matrices. This research direction appears to hold significant potential for the mathematically sound analysis of large-scale, complex-structured data.

Author Contributions

Fatima T. Adilova: Conceptualization, Methodology, Software, Investigation, Writing – original draft, Supervision, Project administration.

Rifqat R. Davronov: Software, Formal analysis, Validation, Writing – review & editing.

Yalkin T. Adilov: Software, Investigation.

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